MATH 220.204, MARCH 29 2019

1. (2018 WT2 Final Exam) For each example below, determine whether $\mathcal{R}$ is a function from A to B.
   - $A = \mathbb{R}, B = \mathbb{Z}, \mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{Z} : x = 3y + 1\}$
   - $A = \mathbb{Q}_{\geq 0}, B = \mathbb{R}, \mathcal{R} = \{(x, y) \in \mathbb{Q}_{\geq 0} \times \mathbb{R} : x^2 = y\}$

2. (2018 WT2 Final) Consider the relation on $\mathbb{Q}$ defined by $a\mathcal{R}b \iff a - b \in \mathbb{Z}$.
   (a) Prove that $\mathcal{R}$ is an equivalence relation.

   (b) Prove that the following statement is false:
   $$\forall a, b \in \mathbb{Q}, (a\mathcal{R}b \implies (\forall q \in \mathbb{Q}, (qa)\mathcal{R}(qb))).$$

   (c) Prove if $a, b \in \mathbb{Q}$ satisfy the property that $\forall q \in \mathbb{Q}, (qa)\mathcal{R}(qb)$, then $a = b$.

3. Let A, B be nonempty sets. Prove that if $|A| \leq |B|$ then $|\mathcal{P}(A)| \leq |\mathcal{P}(B)|$.

4. Let $a, b \in \mathbb{Z}$ be integers such that $a^2 - 3ab + b^2 = 0$. Prove that $a = b = 0$. (Hint: Try mod 3.)

5. In this question, you will construct an explicit bijection to prove that the sets

$$\mathcal{P}(\mathbb{N}) = \{S : S \subseteq \mathbb{N}\} \quad \text{and} \quad (0,1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$$

have the same cardinality. You can prove each step separately, so you may work on later parts first if you prefer.

(a) Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ denote the set of *finite* subsets of $\mathbb{N}$. That is,

$$\mathcal{F} = \{S \subset \mathbb{N} : S \text{ is finite}\}.$$

Prove that $\mathcal{F}$ is countable.

(b) Let $\mathcal{I} = \mathcal{P}(\mathbb{N}) - \mathcal{F}$ be the complement of $\mathcal{F}$. Use the previous part to prove that $\mathcal{I}$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality.

(c) Let $x \in (0,1]$. Define a sequence of positive integers $a_1 < a_2 < a_3 < \dots$ as follows. For every $n \in \mathbb{N}$, a_n is the smallest positive integer such that

$$\frac{1}{2^{a_n}} < x - \frac{1}{2^{a_1}} - \frac{1}{2^{a_2}} - \cdots - \frac{1}{2^{a_{n-1}}}.$$

Prove that this construction is well-defined. That is, prove that for every n, there will always be such a positive integer a_n, and also prove that $a_1 < a_2 < a_3 < \dots$. Conclude that the above procedure defines a function

$$f : (0,1] \to \mathcal{I}, \quad f(x) = \{a_1, a_2, a_3, \dots\}.$$

(d) Prove that if $f(x) = \{a_1, a_2, a_3, \dots\}$ with $a_1 < a_2 < a_3 < \dots$, then

$$x = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \dots.$$

Deduce that f is injective.

(e) Prove that if $x \in (0,1)$ and $x = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \dots$ for some positive integers $a_1 < a_2 < a_3 < \cdots$, then $f(x) = \{a_1, a_2, a_3, \dots\}$. Deduce that f is surjective.